Hale School

Mathematics Specialist

Test 3 --- Term 2 2018

Vectors

/ 45

Name: ______________________________

Instructions:

- **Calculators are allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

All arguments must be given using principal values.

1. **[3, 2 = 5 marks]**

(a) Find the acute angle between $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and the line joining the points

P$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and Q$\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$, giving your answer correct to the nearest degree.

$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$ ✓ Finds $\overrightarrow{PQ}$

$\angle \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$: $\cos\theta = \dfrac{\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{1^2+(-5)^2}\sqrt{2^2+1^2+(-1)^2}}$ ✓ dot product

$\theta = 56°$ ✓ angle to nearest degree.

(b) Find an equation of the plane through Q$\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ and perpendicular to

$\underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, in the form $\underset{\sim}{r} \bullet \underset{\sim}{n} = \rho$.

$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$ ✓ identifies normal vector

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = -16$ ✓ determines $\underset{\sim}{a} \cdot \underset{\sim}{n}$

2. **[5, 5 = 10 marks]**

(a) The shortest distance between the plane $2x-3y+4z=6$ and a parallel plane is 5. Determine the vector equation of the parallel plane.

let $\underset{\sim}{r} = \begin{pmatrix}3\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\4\end{pmatrix}$ be a line $\perp$ to plane.

Parallel plane $\quad d = \left|\lambda\left|\begin{pmatrix}2\\-3\\4\end{pmatrix}\right|\right|$ ✓ calculates $|\underset{\sim}{n}|$

$5 = |\lambda\sqrt{29}|$ ✓ recognises two sol^{n}s.

$\lambda = \pm\frac{5}{\sqrt{29}}$ ✓ uses distance of 5 to evaluate λ

eq^{n} $\parallel$ plane : $\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$

$\underset{\sim}{r}.\begin{pmatrix}2\\-3\\4\end{pmatrix} = \begin{pmatrix}3+2\lambda\\-3\lambda\\4\lambda\end{pmatrix}.\begin{pmatrix}2\\-3\\4\end{pmatrix}$ ✓ evaluates dot product.

$\underset{\sim}{r}.\begin{pmatrix}2\\-3\\4\end{pmatrix} = 6+29\lambda$

$\underset{\sim}{r}.\begin{pmatrix}2\\-3\\4\end{pmatrix} = 6\pm5\sqrt{29}$ ✓ determines correct constant

(b) The plane Π_1 contains the line $\frac{2-x}{3} = -\frac{y}{4} = z+1$ and is parallel to the vector $3\underset{\sim}{i}-2\underset{\sim}{j}$. Find the Cartesian equation of Π_1.

$\lambda = \frac{2-x}{3} \Rightarrow x = 2-3\lambda$ ✓ converts to parametric

$\lambda = -\frac{y}{4} \Rightarrow y = -4\lambda$

$\lambda = z+1 \Rightarrow z = -1+\lambda$

$\therefore \underset{\sim}{r} = \begin{pmatrix}2\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}$ ✓ converts to vector

$\underset{\sim}{n} = \begin{pmatrix}-3\\-4\\1\end{pmatrix} \times \begin{pmatrix}3\\-2\\0\end{pmatrix} = \begin{pmatrix}2\\3\\18\end{pmatrix}$ ✓ evaluates $\underset{\sim}{n}$

$\Pi: \underset{\sim}{r}.\begin{pmatrix}2\\3\\18\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}.\begin{pmatrix}2\\3\\18\end{pmatrix}$ ✓ evaluates $\underset{\sim}{a}.\underset{\sim}{n}$

$\underset{\sim}{r}.\begin{pmatrix}2\\3\\18\end{pmatrix} = -14$

$\Pi: 2x+3y+18z = -14$ ✓ cartesian form

3. **[5, 3 = 8 marks]**

Given the system of equations

$$x+2y+mz=-1$$
$$2x+y-z=3$$
$$mx-2y+z=1$$

(a) Determine the value(s) for m for which there is a unique solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 2 & 1 & -1 & 3 \\ m & -2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 4+m & 0 & -1 & 7 \\ m+1 & 0 & m+1 & 0 \end{array}\right] \begin{array}{l} 2R_2+R_3 \\ R_1+R_3 \end{array}$$

✓ eliminates one variable

$$\left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 4+m & 0 & -1 & 7 \\ 0 & 0 & -(m+1)-(m+1)(4+m) & 7(m+1) \end{array}\right] (m+1)R_2-(4+m)R_3$$

✓ multiplies by $(m+1)$ $(4+m)$ to eliminate second variable.

$$\therefore \left(-m-1-(m^2+5m+4)\right)z = 7(m+1)$$
$$(-m^2-6m-5)\,z = 7(m+1)$$
$$-(m+1)(m+5)\,z = 7(m+1)$$
$$z = \frac{-7(m+1)}{(m+1)(m+5)}$$

✓ expands and factorises

✓ solⁿ for one variable in terms of m

$m \neq -1$, $m \neq -5$ provides a unique solⁿ

✓ correct solⁿ

(b) Determine the value of m for which there are infinite solutions and give geometrical meaning to illustrate this case.

$m=-1$: infinite solutions

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 1 & -1 & 3 \\ -1 & -2 & 1 & 1 \end{array}\right]$$

✓ correct m value

✓ substitutes m value into eqⁿs

- two identical planes (eqⁿ ① & ③ are multiples) and one intersecting plane

✓ correct geometrical interpretation.

4. **[3, 3 = 6 marks]**

$\left|\underset{\sim}{r}-(1,-1,0)\right|=\sqrt{26}$ is the equation of a sphere.

(a) Find the point(s) where the line through $(4,-2,-2)$ and $(6,2,-4)$ meets the sphere.

$$\underset{\sim}{r}=\begin{pmatrix}4\\-2\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\4\\-2\end{pmatrix}$$

$$\left|\begin{pmatrix}4+2\lambda\\-2+4\lambda\\-2-2\lambda\end{pmatrix}-\begin{pmatrix}1\\-1\\0\end{pmatrix}\right|=\sqrt{26}$$ ✓ sub line into plane

$$\left|\begin{pmatrix}3+2\lambda\\-1+4\lambda\\-2-2\lambda\end{pmatrix}\right|=\sqrt{26}$$

$$(3+2\lambda)^2+(-1+4\lambda)^2+(-2-2\lambda)^2=26$$ ✓ solve for λ

$$\lambda=-1,\ 0.5$$

$\therefore$ Meet at $(2,-6,0)$

and $(5,0,-3)$ ✓ points

(b) A plane touches the sphere at the point $(-2,3,1)$. Determine the Cartesian equation of the plane.

$$\underset{\sim}{n}=\begin{pmatrix}-2\\3\\1\end{pmatrix}-\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}-3\\4\\1\end{pmatrix}$$ ✓ normal vector

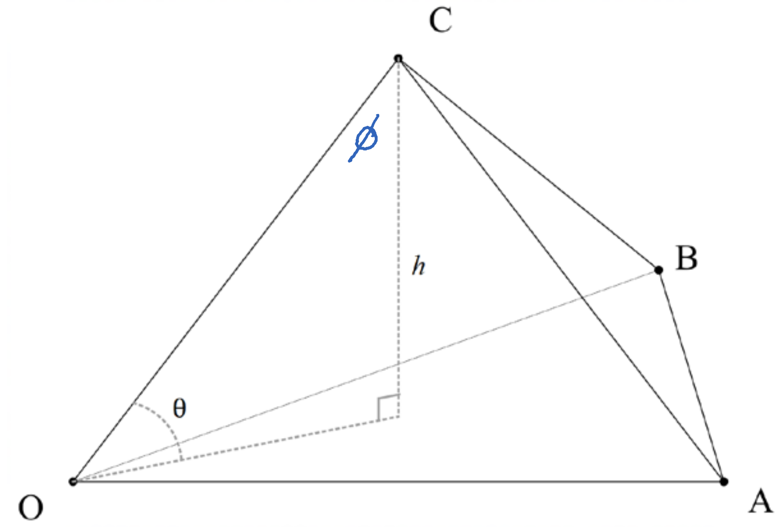
$$\underset{\sim}{r}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=\begin{pmatrix}-3\\4\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\1\end{pmatrix}$$

$$\underset{\sim}{r}\cdot\begin{pmatrix}-3\\4\\1\end{pmatrix}=19$$ ✓ vect eq^n plane

$$-3x+4y+z=19.$$ ✓ cart eq^n plane.

5. **[1, 4 = 5 marks]**

In the tetrahedron shown, $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



(a) Express h in terms of $\underset{\sim}{c}$ and θ.

$h = |\underset{\sim}{c}| \sin\theta$ ✓

(b) Given $V = \frac{1}{3} Base \times h$, show that the volume of the tetrahedron can be found by

$V = \frac{1}{6}|(\underset{\sim}{a} \times \underset{\sim}{b}) \bullet \underset{\sim}{c}|$.

$A(base) = \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}| \sin \angle AOB$ ✓ Determines A(base)

$= \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$

$V = \frac{1}{3} \times \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}| \times |\underset{\sim}{c}| \sin\theta$ ✓ Express for V in terms of $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \theta$

$= \frac{1}{6} |\underset{\sim}{a} \times \underset{\sim}{b}| |\underset{\sim}{c}| \cos\phi$ where ϕ is angle between $\underset{\sim}{a} \times \underset{\sim}{b}$ and $\underset{\sim}{c}$

✓ $\sin\theta = \cos\phi$

✓ illustrates ϕ between $\underset{\sim}{a} \times \underset{\sim}{b}$ and $\underset{\sim}{c}$

$= \frac{1}{6}|(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}|$

✓ Uses dot product

6. [3, 3 = 6 marks]

A curve, called the "witch of Maria Agnesi", is defined by the vector equation

$$\underset{\sim}{r} = \frac{2}{\tan(t)}\underset{\sim}{i} + 2\sin^2(t)\underset{\sim}{j} \ , \ 0 < t < \frac{\pi}{2}.$$

(a) Determine the Cartesian equation of this curve.

$x = \frac{2}{\tan t}$ $\quad y = 2\sin^2 t$

$x^2 = \frac{4\cos^2 t}{\sin^2 t}$

$x^2 = \frac{4(2-2\sin^2 t)}{2\sin^2 t}$

$x^2 = \frac{4(2-y)}{y}$

$x^2 y = 4(2-y)$

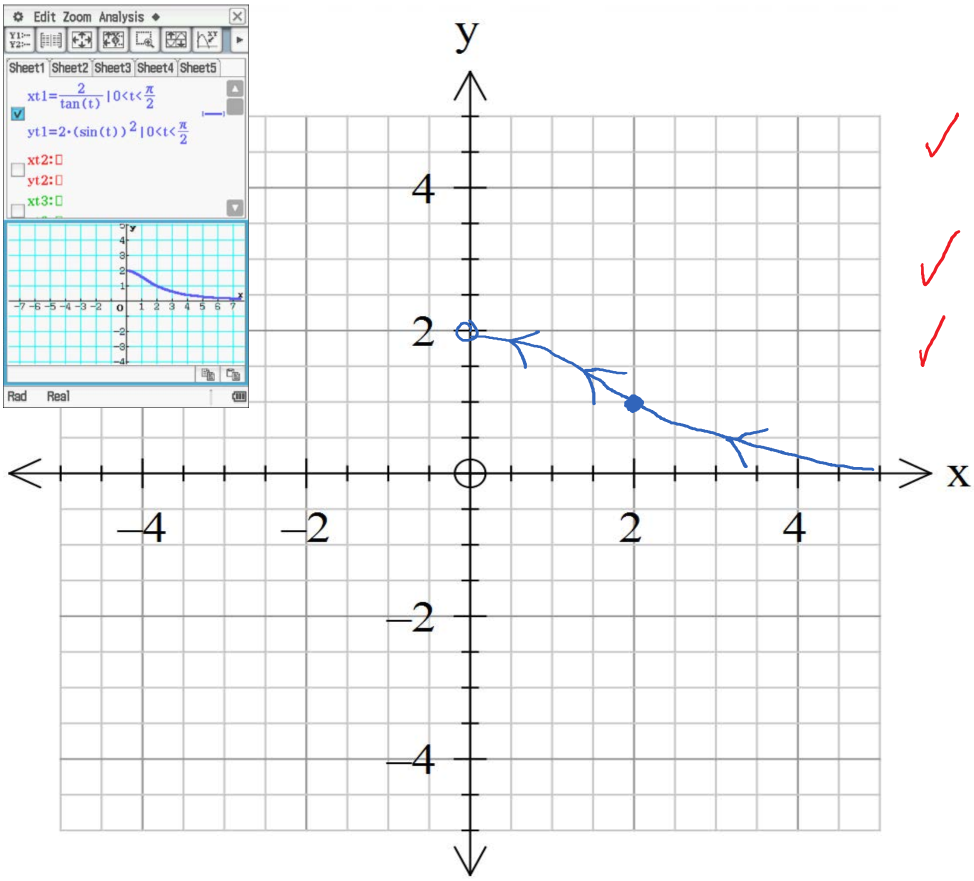
$y = \frac{8}{x^2+4}$

✓ determines parametric

✓ squares x

✓ eliminates the t

(b) Sketch the curve and indicate the direction of motion.



✓ Correct quadrant

✓ Point (2,1)

✓ Direction

7. **[5 marks]**

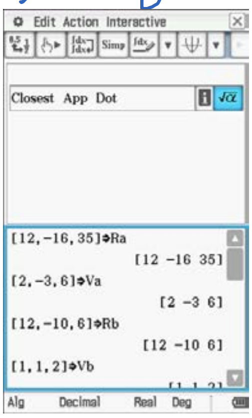
Particle A starts from $(2,-1,5)$ and has a velocity vector $2\underset{\sim}{i}-3\underset{\sim}{j}+6\underset{\sim}{k}$ ms^{-1}. Particle B starts 5 seconds later from $(12,-10,6)$ and has a velocity vector $\underset{\sim}{i}+\underset{\sim}{j}+2\underset{\sim}{k}$ ms^{-1}.

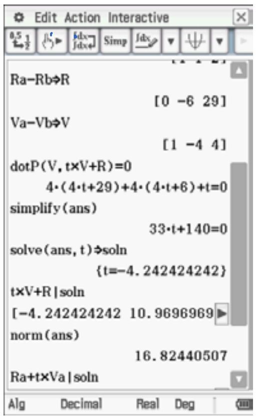
Find the time at which the particles are closest together and the minimum distance.

Let $t=0$ be when Particle B begins moving.

$\therefore\ \underset{\sim}{r}_A = \begin{pmatrix}2\\-1\\5\end{pmatrix} + (t+5)\begin{pmatrix}2\\-3\\6\end{pmatrix} = \begin{pmatrix}12\\-16\\35\end{pmatrix} + t\begin{pmatrix}2\\-3\\6\end{pmatrix}$

$\underset{\sim}{r}_B = \begin{pmatrix}12\\-10\\6\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix}$





Method 1

$\overrightarrow{BA} = \begin{pmatrix}0\\-6\\29\end{pmatrix}$

${}_A\underset{\sim}{V}_B = \begin{pmatrix}1\\-4\\4\end{pmatrix}$

Closest distance when ${}_A\underset{\sim}{V}_B \perp (\overrightarrow{BA} + t \times {}_A\underset{\sim}{V}_B)$

i.e. ${}_A\underset{\sim}{V}_B \cdot (\overrightarrow{BA} + {}_A\underset{\sim}{V}_B \times t) = 0$

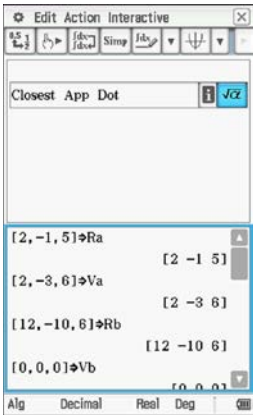
$4(4t+29) + 4(4t+6) + t = 0$

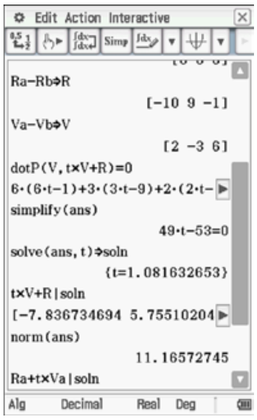
$33t + 140 = 0$

$t = -4.24$ secs

As closest distance happens before B moving, check when B is stationary.

i.e. $\underset{\sim}{r}_A = \begin{pmatrix}2\\-1\\5\end{pmatrix} + t\begin{pmatrix}2\\-3\\6\end{pmatrix}$ $\quad \underset{\sim}{r}_B = \begin{pmatrix}12\\-10\\6\end{pmatrix}$





Closest Approach Calc: $t = 1.08$ s

$d = 11.17$ m

METHOD 2

$d = |\underset{\sim}{r}_B - \underset{\sim}{r}_A| = \left|\begin{pmatrix}0+t\\-6-4t\\29+4t\end{pmatrix}\right|$

$d_{min} = 16.8$ m $\quad t_{min} = -4.24$ secs

check $0 \le t \le 5$, B stationary $\quad d = \left|\begin{pmatrix}-10+2t\\9-3t\\-1+6t\end{pmatrix}\right|$

$\therefore\ t = 1.08$ secs $\quad d = 11.17$ m

✓ determines $\underset{\sim}{r}_A$

✓ determines $\underset{\sim}{r}_B$

✓ expression for ${}_A\underset{\sim}{V}_B \cdot (\overrightarrow{BA} + t\,{}_A\underset{\sim}{V}_B) = 0$

✓ $d_{min} \cdot t_{min}$

✓ Soln when B stationary.

fMin(((2*x-10)^2+(9-3*x)^2+(-1+6*x)^2)^0.5,x)
{MinValue=11.1657274455252,x=1.08163265306122}